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Degenerate elliptic inequalities with critical growth

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Abstract

This article is motivated by the fact that very little is known about variational inequalities of general principal differential operators with critical growth.

The concentration compactness principle of P.L. Lions [P.L. Lions, The concentration compactness principle in the calculus of variation. The limit case I, Rev. Mat. Iberoamericana 1 (1) (1985) 145–201; P.L. Lions, The concentration compactness principle in the calculus of variation. The limit case II, Rev. Mat. Iberoamericana 1 (2) (1985) 45–121] is a widely applied technique in the analysis of Palais–Smale sequences. For critical growth problems involving principal differential operators Laplacian or p-Laplacian, much has been accomplished in recent years, whereas very little has been done for problems involving more general main differential operators since a nonlinearity is observed between the corresponding functional $I(u)$ and measure μ introduced in the concentration compactness method. In this paper, we investigate a Leray–Lions type operator and behaviors of its $(\mathrm{P.S.})_c$ sequence.

Keywords: Critical Sobolev exponent; Variational inequality; Positive solution; Concentration compactness method; $(\mathrm{P.S.})_c$ condition

1. Introduction

For $p > 1$, let $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ denote the usual Sobolev space equipped with the standard norm. We assume that Ω is a bounded domain in $\mathbf{R}^n$ and $\psi : \Omega \to [-\infty, \infty]$ is any

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function in Ω such that $\psi \in W^{1,p}(\Omega)$, $\psi|_{\partial\Omega} \leqslant 0$, and $\psi^+ = \max\{\psi, 0\}$ is positive in a set of positive measure. Define the set

$$\mathrm{K}_\psi := \mathrm{K}_\psi(\Omega) = \{ v \in W_0^{1,p}(\Omega)\colon\ v \geqslant \psi \text{ a.e. in } \Omega \}. \tag{1.1}$$

In this paper, we are interested in the following variational problem involving critical Sobolev growth.

Problem 1.1. Determine $u \in \mathrm{K}_\psi$ such that for all $v \in \mathrm{K}_\psi$

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla(v - u)\, dx \geqslant \lambda \int_\Omega u^{+p^*-1}(v - u)\, dx, \quad \forall v \in \mathrm{K}_\psi, \tag{1.2}$$

where $p^* = \frac{np}{n-p}$, $u^+ = \max\{u, 0\}$, and $\mathcal{A}$-operator is a Leray–Lions type operator [18].

In order to put our results in context, we next present some of the background associated with Problem 1.1.

Variational inequalities arise in the field of mechanics, physics, engineering, control, optimization, equilibrium studies, and nonlinear potential theory [12]. Properties of $\mathcal{A}$-operator have been well documented in the monograph by Heinonen, Kilpeläinen, and Martio [15].

The theory of variational inequalities is closely related to the progress of equations. We now briefly discuss the program on equations with critical Sobolev growth. Semilinear equations involving critical Sobolev exponents, such as

$$-\Delta u = u^{\frac{n+2}{n-2}} + f(x, u),$$

have received great attention ever since the seminal work of Brezis and Nirenberg [4]. The exponent $q = \frac{n+2}{n-2}$ is critical in terms of variational formulation, i.e. $q + 1 = \frac{2n}{n-2}$ is the Sobolev critical exponent. In fact, a well-known nonexistence result by Pohozaev [22] showed that when Ω is starshaped[2] there is no solution to the problem

$$\begin{aligned} -\Delta u &= u^{\frac{n+2}{n-2}} && \text{on } \Omega, \\ u &> 0 && \text{on } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

Many attempts have been made to reverse this situation [4,5,7,21].

Pucci and Serrin [23] extended Pohozaev's nonexistence result to a larger class of variational equations. Their results indicate that an appropriate nonexistence result can be stated for

$$-\Delta_p u = |u|^{q-2} u$$

when $p < n$ and $q \geqslant \frac{np}{n-p} = p^*$, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is a p-Laplacian.

[2] Ω is star-shaped if there exists $x_0 \in \Omega$ such that $(x - x_0) \cdot v \geqslant 0$ for all $x \in \partial\Omega$.

Due to the nonlinear and nonselfadjoint nature of the principal differential operator $\Delta_p u$, many known techniques and results for $p = 2$ are no longer available when $p \neq 2$. It is natural to ask if existence results can be recovered for the p-Laplacian operator involving critical exponents. Many authors have answered this question affirmatively [1,8,14,26].

Variational inequalities involving the $\mathcal{A}$-operator have been studied, see, e.g. [15, Chapter 3], [17] and references therein. Le and Schmitt [17] investigated the existence of positive solutions to variational inequalities involving the $\mathcal{A}$-operator in the subcritical case. We consider here positive solutions with critical nonlinearity. The definition of the $\mathcal{A}$-operator will be made precise now.

Let $\mathrm{F}: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be a *variational kernel* satisfying the following structural assumptions:

1. $\mathrm{F}(x, \xi)$ is measurable for all $\xi \in \mathbf{R}^n$;
2. For a.e. x and some constants $0 < \gamma \leqslant \delta_1 < \infty$,
$$\gamma |\xi|^p \leqslant \mathrm{F}(x, \xi) \leqslant \delta_1 |\xi|^p, \quad \xi \in \mathbf{R}^n,$$
where $1 < p < N$;
3. For a.e. $x \in \mathbf{R}^n$, $\mathrm{F}(x, \xi)$ is strictly convex and differentiable with respect to the second variable ξ. The *strict convexity* of $\xi \to \mathrm{F}(x, \xi)$ means that
$$\mathrm{F}\big(x, t\xi_1 + (1-t)\xi_2\big) < t\mathrm{F}(x, \xi_1) + (1-t)\mathrm{F}(x, \xi_2)$$
whenever $0 < t < 1$ and $\xi_1 \neq \xi_2$.
4. $$\mathrm{F}(x, \lambda\xi) = |\lambda|^p \mathrm{F}(x, \xi), \quad \lambda \in \mathbf{R}.$$

One typical example of $\mathrm{F}(x, \xi)$ is for the following *p-Dirichlet integral* or the *p-energy*
$$\int_\Omega |\nabla u|^p \, dx.$$

With the variational kernel $\mathrm{F}(x, \xi)$ in mind, we define the $\mathcal{A}$-operator next.

Definition 1.2. Let $\mathrm{F}(x, \xi)$ be the variational kernel mentioned above, then we call $\mathcal{A}(x, \xi) = \nabla_\xi \mathrm{F}(x, \xi)$ an *$\mathcal{A}$-operator*.

By [15, Lemma 5.9], $\mathcal{A}(x, \xi)$ satisfies the following Leray–Lions type hypothesis:

5. $\mathcal{A}(x, \xi)$ is a Carathéodory function, i.e. $\mathcal{A}(x, \xi)$ is measurable in x for all $\xi \in \mathbf{R}^n$ and continuous in ξ for a.e. $x \in \mathbf{R}^n$;
6. $$\mathcal{A}(x, \xi) \cdot \xi \geqslant \alpha |\xi|^p;$$
7. $$\mathcal{A}(x, \xi) \leqslant \beta |\xi|^{p-1}$$
for some constants $0 < \alpha \leqslant \beta < \infty$.

8. Instead of the monotonicity property [15, (3.6)], we impose a stronger monotonicity requirement

$$\big(\mathcal{A}(x,\xi_1)-\mathcal{A}(x,\xi_2)\big)\cdot(\xi_1-\xi_2)\geqslant\begin{cases} a|\xi_1-\xi_2|^p & \text{if } p\geqslant 2,\\ \frac{a|\xi_1-\xi_2|^2}{(b+|\xi_1|+|\xi_2|)^{2-p}} & \text{if } 1<p<2,\end{cases}$$

where a and b are positive constants.

The monotonicity assumption (8) can be derived by imposing additional conditions on variational kernel F, for example (see Tolksdorf [25, Lemma 1])

$$\sum_{i,j=1}^{n}\frac{\partial^2 \mathrm{F}(x,\eta)}{\partial\eta_i\partial\eta_j}\cdot\xi_i\xi_j\geqslant\tau\cdot\big(\kappa+|\eta|\big)^{p-2}\cdot\|\xi\|^2$$

for some $\kappa\geqslant 0$ and $\tau>0$. Studying the monotonicity assumption (8) has led to new advances in the theory of $\mathcal{A}$-harmonic equations [16, Chapter 16], quasilinear elliptic equations [25], quasiregular mapping [16, Chapter 15], large distortion in nonlinear elasticity [16, Chapter 15], non-Newtonian Hele–Shaw equations [10,11,13], just to mention a few.

We next briefly describe our approach.

Let $u\in \mathrm{K}_\psi$ be a minimizer of the energy functional

$$I(u)=\int_\Omega F(x,\nabla u)-\frac{\lambda}{p^*}u^{+p^*}\,dx.$$

The properties of variational kernel $F(x,\nabla w)$ imply $I(w)$ is Gateaux and Frechet differentiable. If we take only "one-sided" variations, we obtain

$$\big\langle I'(u),v-u\big\rangle=\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla(v-u)\,dx-\lambda\int_\Omega u^{+p^*-1}(v-u)\,dx\geqslant 0,\quad \forall v\in\mathrm{K}_\psi,$$

where u is a solution to Problem 1.1.

Inspired by the work of [1,3,4,26], we use the variational principle due to Ekeland [9] to find a $(\mathrm{P.S.})_c$ sequence, combined with both the technique developed by Brezis and Nirenberg [4] and the concentration-compactness method (the limit case) of Lions [19,20] to prove that the $(\mathrm{P.S.})_c$ condition is satisfied and eventually recover the loss of compactness in the case of critical Sobolev growth. We establish the minimal positive solution by a supersolution argument.

One major difference occurs between the functional $I(u)$ and measure μ.

In the "concentration compactness" techniques of Lions (for details, see Proposition 2.2), he introduced certain measures μ and ν to handle the energy concentrations. If the principal differential operator is a p-Laplacian, the energy function $I(u)$ becomes

$$I(u)=\int_\Omega\left(\frac{1}{p}|\nabla u|^p-\frac{\lambda}{p^*}u^{+p^*}\right)dx.$$

Applying the linear relationship between measures μ and ν and $I(u)$, some crucial linear equations or inequalities, such as,

$$\mu_j = \nu_j,$$
$$\lim_{n\to\infty} I(u_n) \geqslant I(u) + \left(\frac{1}{p} - \frac{1}{p^*}\right)\sum_j \mu_j = I(u) + \frac{1}{n}\sum_j \mu_j,$$

are obtained (see, for example, [1, Lemma 2.3], [26, Inequality 3.10], [6, Lemma 3.1], [24], [3, Section 4.2], [2, Lemma 7.1]), to recover lack of compactness.

Unfortunately, the functional $I(u)$ in this paper is not linearly related to measure μ. We overcome the nonlinearity between $I(u)$ and μ by nonlinear potential theory, improved trial functions, several well-known inequalities, theory of weak convergence, real analysis, etc.

The paper is organized as follows: in Section 2, we recall some preliminary results. In Section 3, we prove the following theorem:

Theorem 1.3. *Let ψ, K_ψ and $\mathcal{A}$ be as above. If $\frac{\beta}{p^*} < \gamma$, then there exists $\lambda > 0$ such that the variational inequality* (1.1) *has nontrivial nonnegative solutions u.*

Here constants β and γ are those used in condition (2) of the variational kernel F and condition (7) of operator $\mathcal{A}$, respectively. In the case when $\mathcal{A}$ is a p-Laplacian, we have $\beta = 1$ and $\gamma = \frac{1}{p}$. Therefore p-Laplacian satisfies the inequality $\frac{\beta}{p^*} < \gamma$.

Define

$$\lambda^* := \sup\{\lambda > 0\text{: Eq. } (1.2)_\lambda \text{ has nonnegative solutions}\}.$$

In Section 4, under the same conditions as in Theorem 1.3, we establish the following result:

Theorem 1.4. *For every $\lambda \in (0, \lambda^*)$ the variational inequality* (1.1) *has a minimal nonnegative solution $\bar{u}(\lambda)$, which is an increasing function of λ.*

2. Preliminaries

In this section we state the basic concepts and recall some known results. At first we recall the $(\mathrm{P.S.})_c$ condition.

Definition 2.1.

1. We say $u \in \mathrm{K}_\psi$ is a *critical point* of the functional $I(u)$ if

$$\langle I'(u), v - u\rangle \geqslant 0, \quad \forall v \in \mathrm{K}_\psi.$$

2. A sequence $\{u_n\} \subset \mathrm{K}_\psi$ is called a *Palais–Smale sequence at level c* (a $(\mathrm{P.S.})_c$ sequence, in short) if

$$I(u_n) \to c$$

and

$$\left\langle I'(u_n), v - u_n \right\rangle \geqslant \langle z_n, v - u_n \rangle, \quad \forall v \in \mathrm{K}_\psi,$$

where c is a real number and $z_n \to 0$ as $n \to \infty$ in $W^{-1,p'}(\Omega)$.

3. If a $(\mathrm{P.S.})_c$ sequence $\{u_n\}$ implies the existence of a subsequence $\{u_{n_k}\} \subset \{u_n\}$ which converges in K_ψ, we say that I satisfies the $(\mathrm{P.S.})_c$ condition.

Next, we state the concentration-compactness principle of Lions [19,20]. This will be the keystone that enables us to verify that $I(u)$ satisfies the $(\mathrm{P.S.})_c$ condition.

Proposition 2.2. *Let $\{u_n\}$ converge weakly to u in $W_0^{1,p}(\Omega)$ and such that $|u_n|^{p^*}$ converges weakly to a bounded nonnegative measure ν and $|\nabla u_n|^p$ converges weakly to a bounded nonnegative measure μ. Then we have*

1. *There exist some at most countable index set J and two families $\{x_j\}_{j\in J}$ of distinct points in $\overline{\Omega}$, $\{\nu_j\}_{j\in J}$ in $(0,\infty)$ such that*

$$\nu = |u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j};$$

2. $$\mu \geqslant |\nabla u|^p + \sum_{j\in J} \mu_j \delta_{x_j};$$

3. $S\nu_j^{\frac{p}{p^*}} \leqslant \mu_j$, *where S is the best Sobolev constant, i.e.,*

$$S = \inf\left\{ \frac{\|\nabla u\|_{L^p(\Omega)}}{\|u\|_{L^{p^*}(\Omega)}} \colon u \in W_0^{1,p}(\Omega)\backslash\{0\} \right\}. \tag{2.1}$$

Here δ_{x_j} is the Dirac measure at x_j.

The following variational principle is due to Ekeland [9].

Proposition 2.3 *(Ekeland's variational principle [9], Theorem 1.1 bis). Let V be a complete metric space with distance d and $F: V \to \mathbf{R} \cup \{+\infty\}$ be lower semicontinuous, bounded from below, and finite somewhere. For any $\epsilon > 0$, there is some point $v \in V$ with*:

$$F(v) \leqslant \inf_V F + \epsilon \quad \textit{and} \quad F(w) \geqslant F(v) - \epsilon d(v,w), \quad \forall w \in V.$$

As our work uses the $\mathcal{A}$-operator in [15], we refer to that monograph for basic definitions and properties of the $\mathcal{A}$-harmonic operator (see also [17] for the case where the right-hand side is not zero). Let us now introduce some definitions.

Definition 2.4.

1. A function u in $W_0^{1,p}(\Omega)$ is a *weak solution* of the equation

$$-\operatorname{div}\mathcal{A}(x,\nabla u)=f(x),\quad \forall f\in L^{p'}(\Omega),\ p'=\frac{p}{p-1} \tag{2.2}$$

   in Ω if

$$\int_\Omega \mathcal{A}(x,\nabla u)\nabla\varphi\,dx=\int_\Omega f(x)\varphi\,dx$$

   whenever $\varphi\in W_0^{1,p}(\Omega)$.
2. A function u in $W_0^{1,p}(\Omega)$ is a *supersolution* of (2.2) in Ω if

$$-\operatorname{div}\mathcal{A}(x,\nabla u)\geqslant f(x)$$

   weakly in Ω, i.e.

$$\int_\Omega \mathcal{A}(x,\nabla u)\nabla\varphi\,dx\geqslant\int_\Omega f(x)\varphi\,dx$$

   whenever a nonnegative $\varphi\in W_0^{1,p}(\Omega)$.
3. A function $u\in \mathrm{K}_\psi$ that satisfies

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla(v-u)\,dx\geqslant\int_\Omega f(x)(v-u)\,dx,\quad \forall v\in\mathrm{K}_\psi \tag{2.3}$$

   is called a *solution to the variational inequality* in K_ψ.

Le and Schmitt [17, Proposition 3] established the following existence and uniqueness results as an extension of [15, Theorem 3.21].

Lemma 2.5. *Let K_ψ and $\mathcal{A}$ be as above. Then for every $f\in(W_0^{1,p}(\Omega))^*$, the variational inequality* (2.3) *has a unique solution u.*

Since $u+\varphi\in\mathrm{K}_\psi$ for all nonnegative $\varphi\in W_0^{1,p}(\Omega)$, the solution u to variational inequality (2.3) is always a supersolution of (2.2) in Ω. The next comparison lemma indicates that the solution to variational inequality (2.3) is the smallest supersolution of (2.2). The idea behind the proof is similar to [15, Lemmas 3.18 and 3.22].

Lemma 2.6. *Assume that u is a solution to the variational inequality* (2.3) *in K_ψ. If $v\in W_0^{1,p}(\Omega)$ is a supersolution of* (2.2) *in Ω such that* $\min(u,v)\in\mathrm{K}_\psi$*, then $v\geqslant u$ a.e. in Ω.*

Proof. Since $\min(u, v) \in \mathrm{K}_\psi$, the solution u to variational inequality (2.3) must satisfy

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\big(u - \min(u, v)\big)\, dx \leqslant \int_\Omega f(x)\big(u - \min(u, v)\big)\, dx.$$

On the other hand, since $u - \min(u, v) \in W_0^{1,p}(\Omega)$ is nonnegative, the supersolution v must satisfy

$$\int_\Omega \mathcal{A}(x, \nabla v) \cdot \nabla\big(u - \min(u, v)\big)\, dx \geqslant \int_\Omega f(x)\big(u - \min(u, v)\big)\, dx.$$

Then

$$\begin{aligned} 0 &\leqslant \int_\Omega \big(\mathcal{A}(x, \nabla v) - \mathcal{A}(x, \nabla u)\big) \cdot \nabla\big(u - \min(u, v)\big)\, dx \\ &= \int_{\{x \in \Omega:\ v < u\}} \big(\mathcal{A}(x, \nabla v) - \mathcal{A}(x, \nabla u)\big) \cdot \nabla\big(u - \min(u, v)\big)\, dx \\ &= \int_\Omega \big(\mathcal{A}\big(x, \nabla \min(u, v)\big) - \mathcal{A}(x, \nabla u)\big) \cdot \nabla\big(u - \min(u, v)\big)\, dx \leqslant 0. \end{aligned}$$

The monotonicity assumption (8) of the $\mathcal{A}$ operator implies that $u = \min(u, v)$ a.e. in Ω. This completes the proof. □

3. Existence of positive solutions

In this section, we shall prove Theorem 1.3. The proof of Theorem 1.3 will be completed by establishing the validity of a sequence of lemmas.

3.1. The $(P.S.)_c$ condition

We firstly prove a lemma on the qualitative behavior of the $(\mathrm{P.S.})_c$ sequence $\{u_n\}$ for I.

Lemma 3.1. *The* $(\mathrm{P.S.})_c$ *sequence* $\{u_n\}$ *for* I *is bounded in* $W_0^{1,p}(\Omega)$ *if* $\frac{\beta}{p^*} < \gamma$.[3]

Proof. Assume that $\{u_n\} \in \mathrm{K}_\psi$ is a $(\mathrm{P.S.})_c$ sequence, that is,

$$I(u_n) = c + o(1)$$

and

[3] We recall that constants β and γ are those used in condition (2) of the variational kernel F and condition (7) of operator $\mathcal{A}$, respectively.

$$\langle I'(u_n), v - u_n\rangle = \int_\Omega \mathcal{A}(x, \nabla u_n) \cdot \nabla(v - u_n)\,dx - \lambda \int_\Omega {u_n^+}^{p^*-1}(v - u_n)\,dx$$
$$\geqslant \langle z_n, v - u_n\rangle \tag{3.1}$$

for all $v \in \mathrm{K}_\psi$, where c is a real number and $z_n \to 0$ in $W^{-1,p'}(\Omega)$ as $n \to \infty$. Set $v = u_n + u_n^+$ in Eq. (3.1) (clearly $u_n + u_n^+ \in \mathrm{K}_\psi$), we have[4]

$$\int_\Omega \mathcal{A}(x, \nabla u_n) \cdot \nabla u_n^+\,dx \geqslant \lambda \int_\Omega {u_n^+}^{p^*}\,dx + \langle z_n, u_n^+\rangle.$$

Letting $\Omega^+(u_n) = \{x \in \Omega\colon\ u_n(x) \geqslant 0\}$ and $\Omega^-(u_n) = \{x \in \Omega\colon\ u_n(x) < 0\}$, we can rewrite the above inequality as

$$\int_{\Omega^+(u_n)} \mathcal{A}\big(x, \nabla u_n^+\big) \cdot \nabla u_n^+\,dx \geqslant \lambda \int_{\Omega^+(u_n)} {u_n^+}^{p^*}\,dx + \langle z_n, u_n^+\rangle,$$

since

$$\nabla u_n^+ = \begin{cases} \nabla u(x) & \text{if } u(x) \geqslant 0,\\ 0 & \text{if } u(x) < 0.\end{cases}$$

Using property (7) of the $\mathcal{A}$ operator, we have

$$\beta \int_{\Omega^+(u_n)} \big|\nabla u_n^+\big|^p\,dx \geqslant \lambda \int_{\Omega^+(u_n)} {u_n^+}^{p^*}\,dx + \langle z_n, u_n^+\rangle. \tag{3.2}$$

Next we choose $v = u_n - u_n^-$ in Eq. (3.1), where $u_n^- = \min\{u_n, 0\}$ (clearly $u_n - u_n^- \in \mathrm{K}_\psi$), we have

$$\int_{\Omega^-(u_n)} \mathcal{A}(x, \nabla u_n^-) \cdot \nabla u_n^-\,dx \leqslant \langle z_n, u_n^-\rangle.$$

By property (6) of operator $\mathcal{A}$, we have

$$\alpha \int_{\Omega^-(u_n)} |\nabla u_n^-|^p\,dx \leqslant \langle z_n, u_n^-\rangle. \tag{3.3}$$

This implies that $|\nabla u_n^-| \to 0$ a.e. in Ω. It follows from Poincaré inequality that $|u_n^-| \to 0$ almost everywhere.

From estimate (2) in the variational kernel F, estimates (3.2) and (3.3), we have

[4] We recall that $u_n^+ = \max\{u_n, 0\}$.

$$o(1)+c=I(u_n)\geqslant \gamma\int_{\Omega}|\nabla u_n|^p\,dx-\frac{\lambda}{p^*}\int_{\Omega^+(u_n)}u_n^{+p^*}\,dx$$

$$\geqslant\left(\gamma-\frac{\beta}{p^*}\right)\int_{\Omega}|\nabla u_n|^p\,dx+\langle z_n,u_n^+\rangle$$

$$\geqslant\left(\gamma-\frac{\beta}{p^*}\right)\|u_n\|^p_{W_0^{1,p}(\Omega)}-C\|z_n\|_{W^{-1,p'}(\Omega)}\|u_n\|_{W_0^{1,p}(\Omega)}.$$

This concludes the proof. □

The concentration-compactness principle 2.2 and previous lemma allow us to verify a $(\mathrm{P.S.})_c$ condition.

Proposition 3.2. *Let $\{u_n\}$ be the $(\mathrm{P.S.})_c$ sequence for I, defined by 2.1, $\frac{\beta}{p^*}<\gamma$, and ψ^+ be positive somewhere. Then there exists a subsequence $\{u_{n_k}\}\subset\{u_n\}$ such that*

$$u_{n_k}\to u\quad \textit{strongly in } W_0^{1,p}(\Omega),\tag{3.4}$$

$$\mathcal{A}(x,\nabla u_{n_k})\rightharpoonup\mathcal{A}(x,\nabla u)\quad \textit{weakly in } \big(L^p(\Omega)\big)^*,\tag{3.5}$$

where $i=1,2,\ldots,N$, $(L^p(\Omega))^*$ is the dual space of $L^p(\Omega)$, and $u\in \mathrm{K}_\psi$.

The proof of this result is rather long. To simplify the presentation, we split the proof in various steps. First, by the concentration-compactness principle 2.2, Sobolev embedding, Lemma 3.1, and inequality (3.3), we may assume, up to a subsequence, that

$$u_{n_k}\rightharpoonup u\quad \text{weakly in } W_0^{1,p}(\Omega),\tag{3.6}$$

$$u_{n_k}\to u\quad \text{in } L^q(\Omega),\ 1<q<p^*,\tag{3.7}$$

$$u_{n_k}\to u\quad \text{a.e. in } \Omega,\tag{3.8}$$

$$|\nabla u_{n_k}|^p\rightharpoonup\mu\geqslant|\nabla u|^p+\sum_{j\in J}\mu_j\delta_{x_j},\tag{3.9}$$

$$u_{n_k}^{+\,p^*}\rightharpoonup\nu=u^{+p^*}+\sum_{j\in J}\nu_j\delta_{x_j},\tag{3.10}$$

where μ and ν are bounded nonnegative measures. The convergence (3.8) implies that $u\in\mathrm{K}_\psi$.

Next we show that the set J is finite.

Lemma 3.3. *There exist at most a finite number of points x_j on Ω.*

Proof. From Proposition 2.2,

$$\mu_j\geqslant S\nu_j^{\frac{p}{p^*}}.\tag{3.11}$$

We define $\eta \in C_0^\infty(\mathbf{R}^n)$ such that

$$0 \leqslant \eta \leqslant 1, \quad \eta \equiv 1 \text{ on } B\left(0, \frac{1}{2}\right), \quad \eta \equiv 0 \text{ on } \mathbf{R}^n \setminus B(0,1), \quad \text{and} \quad |\nabla \eta| < C,$$

where C is some constant. Denote $\eta_\epsilon(x) = \eta_\epsilon^j(x) = \eta(\frac{x-x_j}{2\epsilon})$, for $\epsilon > 0$, $x \in \Omega$, and then $|\nabla \eta_\epsilon| < C/\epsilon$. It follows from Lemma 3.1 that $\eta_\epsilon u_{n_k}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. Setting

$$v = u_{n_k} - \eta_\epsilon(u_{n_k} - \psi) = (1-\eta_\epsilon)u_{n_k} + \eta_\epsilon \psi \geqslant \psi \in \mathrm{K}_\psi,$$

in Eq. (3.1), we have

$$\begin{aligned}
&\int_\Omega \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla \eta_\epsilon (u_{n_k} - \psi)\, dx + \int_\Omega \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla (u_{n_k} - \psi)\eta_\epsilon \, dx \\
&\quad \leqslant \lambda \int_\Omega \eta_\epsilon u_{n_k}^{+\,p^*-1}(u_{n_k} - \psi)\, dx + o(1).
\end{aligned}$$

Now, from (3.10), letting $k \to \infty$, we have

$$\begin{aligned}
&\lim_{k\to\infty}\left[\int_\Omega \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla \eta_\epsilon (u_{n_k} - \psi)\, dx + \int_\Omega \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla (u_{n_k} - \psi)\eta_\epsilon \, dx\right] \\
&\quad \leqslant \lambda \int_\Omega \eta_\epsilon \left(u^{+\,p^*} + \sum_{j\in J} \nu_j \delta_{x_j}\right) dx - \lambda \lim_{k\to\infty} \int_\Omega \eta_\epsilon u_{n_k}^{+\,p^*-1} \psi \, dx.
\end{aligned}$$

Invoking Sobolev embedding, Hölder's inequality, estimate (6) of $\mathcal{A}$-operator, and (3.10), we get

$$\begin{aligned}
0 &\leqslant \lim_{k\to\infty}\left|\int_\Omega \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla \eta_\epsilon (u_{n_k} - \psi)\, dx\right| \\
&\leqslant \lim_{k\to\infty} \beta \int_\Omega |\nabla u_{n_k}|^{p-1} |\nabla \eta_\epsilon| \left|(u_{n_k} - \psi)\right| dx \\
&\leqslant \lim_{k\to\infty} \beta \left(\int_\Omega |\nabla u_{n_k}|^p \, dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla \eta_\epsilon|^p \left|(u_{n_k} - \psi)\right|^p dx\right)^{\frac{1}{p}} \\
&= \lim_{k\to\infty} C \left(\int_{B(x_j, 2\epsilon)} |\nabla \eta_\epsilon|^p \left|(u_{n_k} - \psi)\right|^p dx\right)^{\frac{1}{p}} \\
&\leqslant \lim_{k\to\infty} C \left(\int_{B(x_j, 2\epsilon)} |\nabla \eta_\epsilon|^n \, dx\right)^{\frac{1}{n}} \left(\int_{B(x_j, 2\epsilon)\setminus B(x_j, \epsilon)} \left|(u_{n_k} - \psi)\right|^{p^*} dx\right)^{\frac{1}{p^*}}
\end{aligned}$$

$$\leqslant \lim_{k\to\infty} C\biggl(\int\limits_{B(x_j,2\epsilon)\setminus B(x_j,\epsilon)} \bigl|(u_{n_k}-\psi)\bigr|^{p^*}\,dx\biggr)^{\frac{1}{p^*}}.$$

Note that constant C may be different at different occurrences.

We claim that

$$\lim_{k\to\infty} C\biggl(\int\limits_{B(x_j,2\epsilon)\setminus B(x_j,\epsilon)} \bigl|(u_{n_k}-\psi)\bigr|^{p^*}\,dx\biggr)^{\frac{1}{p^*}} \to 0$$

as $\epsilon \to 0$.

1. We first treat the case when x_j is an isolated point of $(x_j)_{j\in J}$. Then there exists $\epsilon > 0$ such that $B(x_j, 2\epsilon) \cap (x_j)_{j\in J} = \{x_j\}$. Therefore

$$\begin{aligned}
&\lim_{k\to\infty} C\biggl(\int\limits_{B(x_j,2\epsilon)\setminus B(x_j,\epsilon)} \bigl|(u_{n_k}-\psi)\bigr|^{p^*}\,dx\biggr)^{\frac{1}{p^*}} \\
&\quad = C\biggl(\int\limits_{B(x_j,2\epsilon)\setminus B(x_j,\epsilon)} \bigl|(u-\psi)\bigr|^{p^*}\,dx\biggr)^{\frac{1}{p^*}} \\
&\quad \leqslant C\biggl(\int\limits_{B(x_j,2\epsilon)} \bigl|(u-\psi)\bigr|^{p^*}\,dx\biggr)^{\frac{1}{p^*}} \to 0 \quad \text{as } \epsilon \to 0.
\end{aligned}$$

2. [5]We now consider the case when x_j is an accumulation point of $(x_j)_{j\in J}$. Then

$$B(x_j, 2\epsilon) \cap (x_j)_{j\in J} = \bigl\{x_{j_{i(2\epsilon)}} \in B(x_j, 2\epsilon)\colon\ j_{i(2\epsilon)} \in J\bigr\}$$

is an infinite subset of $(x_j)_{j\in J}$ for any $\epsilon > 0$. It follows that

$$\begin{aligned}
&\lim_{k\to\infty} C\biggl(\int\limits_{B(x_j,2\epsilon)\setminus B(x_j,\epsilon)} \bigl|(u_{n_k}-\psi)\bigr|^{p^*}\,dx\biggr)^{\frac{1}{p^*}} \\
&\quad = C\biggl(\int\limits_{B(x_j,2\epsilon)\setminus B(x_j,\epsilon)} \bigl|(u-\psi)\bigr|^{p^*}\,dx + \sum_{j_{i(2\epsilon)}} \nu_{j_{i(2\epsilon)}} - \sum_{j_{i(\epsilon)}} \nu_{j_{i(\epsilon)}}\biggr)^{\frac{1}{p^*}}.
\end{aligned}$$

Since $\sum_{j\in J} \nu_j < \infty$, we have

$$\sum_{i(2\epsilon)} \nu_{j_{i(2\epsilon)}} - \nu_j \to 0$$

as $\epsilon \to 0$. Our claim is proved.

[5] Similar argument can also be found in [3].

Similarly,

$$
\begin{aligned}
0 &\leqslant \lim_{k\to\infty}\left|\int\limits_{\Omega} \mathcal{A}(x,\nabla u_{n_k})\cdot\nabla\psi\,\eta_\epsilon\,dx\right| \\
&\leqslant \lim_{k\to\infty}\beta\int\limits_{\Omega}|\nabla u_{n_k}|^{p-1}|\nabla\psi||\eta_\epsilon|\,dx \\
&\leqslant \lim_{k\to\infty}\beta\Big(\int\limits_{\Omega}|\nabla u_{n_k}|^p\,dx\Big)^{\frac{p-1}{p}}\Big(\int\limits_{\Omega}|\eta_\epsilon\nabla\psi|^p\,dx\Big)^{\frac{1}{p}} \\
&\leqslant C\Big(\int\limits_{B(x_j,2\epsilon)}|\nabla\psi|^p\,dx\Big)^{\frac{1}{p}}\to 0\quad\text{as } \epsilon\to 0,
\end{aligned}
$$

and

$$
\lim_{k\to\infty}\int\limits_{\Omega}\eta_\epsilon {u_{n_k}^+}^{p^*-1}\psi\,dx=\int\limits_{\Omega}\eta_\epsilon {u^+}^{p^*-1}\psi\,dx=\int\limits_{B(x_j,2\epsilon)}\eta_\epsilon {u^+}^{p^*-1}\psi\,dx\to 0
$$

as $\epsilon\to 0$. Therefore, by (3.9), assumption (6) of $\mathcal{A}$-operator, and taking $k\to\infty$, we obtain

$$
\alpha\int\limits_{\Omega}\eta_\epsilon\Big(|\nabla u|^p+\sum_{j\in J}\mu_j\delta_{x_j}\Big)\,dx\leqslant\lambda\int\limits_{\Omega}\eta_\epsilon\Big({u^+}^{p^*}+\sum_{j\in J}\nu_j\delta_{x_j}\Big)\,dx+o(\epsilon).
$$

Repeating the same argument as we did in p. 452 and passing to the limit, we get

$$
\alpha\mu_j\leqslant\lambda\nu_j.
$$

Thus, from (3.11), we get $\frac{\lambda}{\alpha}\nu_j\geqslant S\nu_j^{\frac{p}{p^*}}$, and consequently, $\nu_j\geqslant(\frac{\alpha S}{\lambda})^{N/p}$. Since $\sum_{j\in J}\nu_j<\infty$, we conclude the proof of Lemma 3.3. □

Using the monotonicity condition (8) of the $\mathcal{A}$-operator, we may show

Lemma 3.4.

$$
\begin{aligned}
&C\int\limits_{\Omega}\big(\mathcal{A}(x,\nabla u_{n_k})-\mathcal{A}(x,\nabla u)\big)\cdot(\nabla u_{n_k}-\nabla u)\,dx \\
&\quad\geqslant\begin{cases}\int_\Omega|\nabla u_{n_k}-\nabla u|^p\,dx & \text{if } p\geqslant 2,\\ \big(\int_\Omega|\nabla u_{n_k}-\nabla u|^p\,dx\big)^{2/p} & \text{if } 1<p<2.\end{cases}
\end{aligned}
$$

Proof. If $p \geqslant 2$, the lemma is a direct result of monotonicity assumption (8).

When $1 < p < 2$, by Hölder's inequality and monotonicity assumption (8), we have

$$a \int_\Omega |\nabla u_{n_k} - \nabla u|^p \, dx$$
$$\leqslant \int_\Omega \big[\big(\mathcal{A}(x, \nabla u_{n_k}) - \mathcal{A}(x, \nabla u)\big) \cdot (\nabla u_{n_k} - \nabla u)\big]^{\frac{p}{2}} \big(b + |\nabla u_{n_k}| + |\nabla u|\big)^{\frac{(2-p)p}{2}} \, dx$$
$$\leqslant \bigg[\int_\Omega \big(\mathcal{A}(x, \nabla u_{n_k}) - \mathcal{A}(x, \nabla u)\big) \cdot (\nabla u_{n_k} - \nabla u) \, dx\bigg]^{\frac{p}{2}} \bigg[\int_\Omega \big(b + |\nabla u_{n_k}| + |\nabla u|\big)^p \, dx\bigg]^{1-\frac{p}{2}}.$$

The lemma follows from Lemma 3.1. □

Lemma 3.5.

$$\nabla u_{n_k} \to \nabla u \quad \textit{strongly in } L^p(\Omega) \tag{3.12}$$

as $n \to \infty$.

Proof. From Lemma 3.4, it is sufficient to prove

$$\int_\Omega \big(\mathcal{A}(x, \nabla u_{n_k}) - \mathcal{A}(x, \nabla u)\big) \cdot (\nabla u_{n_k} - \nabla u) \, dx \to 0. \tag{3.13}$$

This will be discussed for two different cases (i) $J = \{1, 2, \dots, m\}$, (ii) $J = \emptyset$.

Case (i). $J = \{1, 2, \dots, m\}$. Take R sufficiently large such that $\overline{\Omega} \subset \mathbf{B}(0, R)$. Define $\zeta_\epsilon = 1 - \sum_{j=1}^m \eta_\epsilon^j(x) - \zeta$, where $\eta_\epsilon^j(x)$ is defined in Lemma 3.3 and $\zeta \in C^\infty(\mathbf{R}^n)$ is a smooth cutoff function satisfying

$$0 \leqslant \zeta \leqslant 1, \quad \zeta(x) = 0 \quad \text{on } \mathbf{B}(0, R), \quad \zeta(x) = 1 \quad \text{on } \mathbf{R}^n \setminus \mathbf{B}(0, 2R).$$

Let ϵ sufficiently small such that $\mathbf{B}(x_i, 2\epsilon) \cap \mathbf{B}(x_j, 2\epsilon) = \emptyset$ when $i \neq j$ and $i, j = 1, 2, \dots, m$. Clearly $\zeta_\epsilon \in C^\infty(\mathbf{R}^n)$, $\operatorname{supp} \zeta_\epsilon \subset \mathbf{B}(0, 2R)$, $\zeta_\epsilon(x) = 0$ if $x \in \bigcup_{j=1}^m \mathbf{B}(x_j, \epsilon)$, $\zeta_\epsilon(x) = 1$ if $x \in (\mathbf{R}^n \setminus \bigcup_{j=1}^m \mathbf{B}(x_j, 2\epsilon)) \cap \mathbf{B}(0, R)$. Choosing $v = u_{n_k} - \zeta_\epsilon(u_{n_k} - u) = (1 - \zeta_\epsilon)u_{n_k} + \zeta_\epsilon u \geqslant \psi \in \mathrm{K}_\psi$ in (3.1), we have

$$\int_\Omega \zeta_\epsilon \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla u_{n_k} \, dx + \int_\Omega u_{n_k} \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla \zeta_\epsilon \, dx$$
$$- \int_\Omega \zeta_\epsilon \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla u \, dx - \int_\Omega u \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla \zeta_\epsilon \, dx$$
$$\leqslant \lambda \int_\Omega \zeta_\epsilon {u_{n_k}^+}^{p^*} \, dx - \lambda \int_\Omega \zeta_\epsilon {u_{n_k}^+}^{p^*-1} u \, dx + o(1).$$

From (3.10), the construction of the cutoff function ζ_ϵ and (3.7), we obtain

$$\begin{aligned}
\lim_{k\to\infty}\biggl(\int_\Omega \zeta_\epsilon {u_{n_k}^+}^{p^*}\,dx - \int_\Omega \zeta_\epsilon {u_{n_k}^+}^{p^*-1}u\,dx\biggr)
&= \int_\Omega \zeta_\epsilon\Bigl({u^+}^{p^*} + \sum_{j\in J}\nu_j\delta_{x_j}\Bigr)\,dx - \int_\Omega \zeta_\epsilon {u^+}^{p^*}\,dx \\
&= \int_\Omega \zeta_\epsilon \sum_{j\in J}\nu_j\delta_{x_j}\,dx \\
&= \sum_{j\in J}\nu_j\zeta_\epsilon(x_j) = 0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\lim_{k\to\infty}\biggl[\int_\Omega \zeta_\epsilon\mathcal{A}(x,\nabla u_{n_k})\cdot\nabla u_{n_k}\,dx - \int_\Omega \zeta_\epsilon\mathcal{A}(x,\nabla u_{n_k})\cdot\nabla u\,dx\biggr] \\
&\quad\leqslant \lim_{k\to\infty}\biggl[\int_\Omega u\mathcal{A}(x,\nabla u_{n_k})\cdot\nabla\zeta_\epsilon\,dx - \int_\Omega u_{n_k}\mathcal{A}(x,\nabla u_{n_k})\cdot\nabla\zeta_\epsilon\,dx\biggr].
\end{aligned}\tag{3.14}$$

Considering

$$\begin{aligned}
0 &\leqslant \int_\Omega \zeta_\epsilon\bigl(\mathcal{A}(x,\nabla u_{n_k}) - \mathcal{A}(x,\nabla u)\bigr)\cdot(\nabla u_{n_k} - \nabla u)\,dx \\
&= \int_\Omega \zeta_\epsilon\mathcal{A}(x,\nabla u_{n_k})\cdot\nabla u_{n_k}\,dx + \int_\Omega \zeta_\epsilon\mathcal{A}(x,\nabla u)\cdot\nabla u\,dx \\
&\quad - \int_\Omega \zeta_\epsilon\mathcal{A}(x,\nabla u)\cdot\nabla u_{n_k}\,dx - \int_\Omega \zeta_\epsilon\mathcal{A}(x,\nabla u_{n_k})\cdot\nabla u\,dx.
\end{aligned}\tag{3.15}$$

Note that $\nabla(u_{n_k} - u) \rightharpoonup 0$ in $L^p(\Omega)$ and $\mathcal{A}(x,\nabla u) \in L^{p'}(\Omega)$, we have

$$\int_\Omega \zeta_\epsilon\mathcal{A}(x,\nabla u)\cdot\nabla(u - u_{n_k})\,dx \to 0 \quad \text{as } k\to\infty.$$

For any given $\epsilon > 0$, by (3.14),

$$\begin{aligned}
0 &\leqslant \lim_{k\to\infty}\int_\Omega \zeta_\epsilon\bigl(\mathcal{A}(x,\nabla u_{n_k}) - \mathcal{A}(x,\nabla u)\bigr)\cdot(\nabla u_{n_k} - \nabla u)\,dx \\
&= \lim_{k\to\infty}\biggl[\int_\Omega \zeta_\epsilon\mathcal{A}(x,\nabla u_{n_k})\cdot\nabla u_{n_k}\,dx - \int_\Omega \zeta_\epsilon\mathcal{A}(x,\nabla u_{n_k})\cdot\nabla u\,dx\biggr]
\end{aligned}$$

$$\leqslant \lim_{k\to\infty}\int_\Omega (u-u_{n_k})\mathcal{A}(x,\nabla u_{n_k})\cdot\nabla\zeta_\epsilon\,dx$$

$$\leqslant \lim_{k\to\infty}\beta\|\nabla\zeta_\epsilon\|_{L^\infty(\Omega)}\|u-u_{n_k}\|_{L^p(\Omega)}\|\nabla u_{n_k}\|_{L^p(\Omega)}^{p-1}.$$

Now from (3.7), boundedness of ∇u_{n_k}, we have for any given $\epsilon>0$

$$\lim_{k\to\infty}\int_\Omega \zeta_\epsilon\big(\mathcal{A}(x,\nabla u_{n_k})-\mathcal{A}(x,\nabla u)\big)\cdot(\nabla u_{n_k}-\nabla u)\,dx=0.$$

We set $\Omega_{\epsilon_0}:=\bigcap_{j=1}^m\{x\in\Omega\colon \operatorname{dist}(x,x_j)>\epsilon_0\}$ for $\epsilon_0>0$. It follows that

$$0\leqslant\int_{\Omega_{\epsilon_0}}\big(\mathcal{A}(x,\nabla u_{n_k})-\mathcal{A}(x,\nabla u)\big)\cdot(\nabla u_{n_k}-\nabla u)\,dx$$

$$\leqslant\lim_{\epsilon\to0}\lim_{k\to\infty}\int_\Omega \zeta_\epsilon\big(\mathcal{A}(x,\nabla u_{n_k})-\mathcal{A}(x,\nabla u)\big)\cdot(\nabla u_{n_k}-\nabla u)\,dx=0.$$

Taking $\epsilon_0\to0$, we obtain

$$\int_\Omega\big(\mathcal{A}(x,\nabla u_{n_k})-\mathcal{A}(x,\nabla u)\big)\cdot(\nabla u_{n_k}-\nabla u)\,dx\to0.$$

Case (ii). $J=\emptyset$. The argument is similar but simpler than the previous case since $\epsilon_0=0$ and $\zeta_\epsilon\equiv1$ in $\overline{\Omega}$. Lemma 3.5 is proved. □

As a direct consequence of Lemma 3.5, we have

Corollary 3.6. *The sequence* $\{u_{n_k}\}\in \mathrm{K}_\psi$ *possesses a subsequence* (*still label as* $\{u_{n_k}\}$) *satisfying*

$$\frac{\partial u_{n_k}}{\partial x_i}\to\frac{\partial u}{\partial x_i}\quad \textit{a.e. in } \Omega. \tag{3.16}$$

Finally, we conclude the proof of Proposition 3.2. By Corollary 3.6, we have

$$\mathcal{A}(x,\nabla u_{n_k})\to\mathcal{A}(x,\nabla u)\quad\text{a.e. in }\Omega.$$

Egorov's Theorem implies that $\forall\epsilon>0$, there is a closed subset Ω_ϵ of Ω such that $|\Omega-\Omega_\epsilon|<\epsilon$ and $\mathcal{A}(x,\nabla u_{n_k})\to\mathcal{A}(x,\nabla u)$ uniformly on Ω_ϵ. Then $\forall\phi\in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega\big(\mathcal{A}(x,\nabla u_{n_k})-\mathcal{A}(x,\nabla u)\big)\phi\,dx$$

$$=\int_{\Omega_\epsilon}+\int_{\Omega-\Omega_\epsilon}\big(\mathcal{A}(x,\nabla u_{n_k})-\mathcal{A}(x,\nabla u)\big)\phi\,dx$$

$$\leqslant \epsilon \int\limits_{\Omega_\epsilon} |\phi|\,dx + \beta\big(\|\nabla u_{n_k}\|_{L^p(\Omega-\Omega_\epsilon)}^{(p-1)} + \|\nabla u\|_{L^p(\Omega-\Omega_\epsilon)}^{(p-1)}\big)\|\phi\|_{L^p(\Omega-\Omega_\epsilon)}.$$

The weak convergence (3.5) follows by letting $\epsilon \to 0$ and $k \to \infty$ and applying the absolute continuity of the Lebesgue Integral with respect to Lebesgue measure. Proposition 3.2 is proved.

3.2. Existence of nonnegative solutions

Proposition 3.7. *Any* (P.S.)$_c$ *sequence possesses a subsequence converging weakly in* $W_0^{1,p}(\Omega)$ *to a nonnegative solution of Problem* 1.1.

Proof. We first argue that $J = \emptyset$. Invoking the Sobolev Embedding Theorem and Proposition 3.2 leads to

$$u_{n_k} \to u \quad \text{in } L^{p^*}(\Omega). \tag{3.17}$$

Radon–Riesz Theorem suggests that $\|u_{n_k}\|_{L^{p^*}(\Omega)} \to \|u\|_{L^{p^*}(\Omega)}$. Therefore, by inequality (3.3),

$$\big\|u_{n_k}^+\big\|_{L^{p^*}(\Omega)} \to \big\|u^+\big\|_{L^{p^*}(\Omega)}.$$

Combining with (3.10), we get $J = \emptyset$.

Now, we show that u is a solution of Problem 1.1.

Denote $u \wedge v = \min\{u, v\}$ and $v_{n_k} = u_{n_k} - u$. Let $v = u_{n_k} + (v_{n_k} - \psi^+)^+$ (clearly $v \geqslant u_{n_k} \geqslant \psi$) in (3.1). Since $(v_{n_k} - \psi^+)^+ = v_{n_k} - v_{n_k} \wedge \psi^+$, (3.1) becomes

$$\begin{aligned}\int\limits_\Omega \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla v_{n_k}\,dx - \int\limits_\Omega \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla\big(v_{n_k} \wedge \psi^+\big)\,dx \\ \geqslant \lambda \int\limits_\Omega {u_{n_k}^+}^{p^*-1}\big(v_{n_k} - v_{n_k} \wedge \psi^+\big)\,dx + o(1).\end{aligned} \tag{3.18}$$

Applying Hölder's inequality and (3.17), we obtain

$$\left|\int\limits_\Omega {u_{n_k}^+}^{p^*-1} v_{n_k} \wedge \psi^+\,dx\right| \leqslant C\bigg(\int\limits_\Omega |v_{n_k}|^{p^*}\,dx\bigg)^{1/p^*} \to 0.$$

We can rewrite the second term on the left-hand side of (3.18) as

$$\begin{aligned}&\int\limits_\Omega \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla\big(v_{n_k} \wedge \psi^+\big)\,dx \\ &\quad = \int\limits_{v_{n_k} \leqslant \psi^+} \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla v_{n_k}\,dx + \int\limits_{v_{n_k} > \psi^+} \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla \psi^+\,dx\end{aligned}$$

$\|u_{n_k} - u\|_{L^p(\Omega)} \to 0$ implies the measure $m\{x \in \Omega\colon\ v_{n_k} > \psi^+\} \to 0$ as $k \to \infty$. Therefore, applying Hölder's inequality and the absolute continuity of the Lebesgue Integral with respect to Lebesgue measure, we get

$$\left| \int\limits_{v_{n_k} > \psi^+} \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla \psi^+ \, dx \right| \leqslant C \biggl(\int\limits_{v_{n_k} > \psi^+} |\nabla \psi|^p \, dx \biggr)^{1/p} \to 0.$$

By hypothesis (7) of operator $\mathcal{A}$, Hölder's inequality, and Proposition 3.2, we obtain

$$\begin{aligned} \left| \int\limits_{v_{n_k} \leqslant \psi^+} \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla v_{n_k} \, dx \right| &\leqslant C \biggl(\int\limits_{v_{n_k} \leqslant \psi^+} |\nabla v_{n_k}|^p \, dx \biggr)^{1/p} \\ &\leqslant C \biggl(\int\limits_{\Omega} |\nabla v_{n_k}|^p \, dx \biggr)^{1/p} \to 0. \end{aligned}$$

Therefore

$$\int\limits_{\Omega} \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla \bigl(v_{n_k} \wedge \psi^+ \bigr) \, dx \to 0.$$

It follows that

$$\int\limits_{\Omega} \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla v_{n_k} \, dx \geqslant \lambda \int\limits_{\Omega} {u_{n_k}^+}^{p^*-1} v_{n_k} \, dx + o(1),$$

that is,

$$\begin{aligned} &\int\limits_{\Omega} \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla u_{n_k} \, dx - \lambda \int\limits_{\Omega} {u_{n_k}^+}^{p^*} \, dx \\ &\quad \geqslant \int\limits_{\Omega} \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla u \, dx - \lambda \int\limits_{\Omega} {u_{n_k}^+}^{p^*-1} u \, dx + o(1) \end{aligned}$$

(3.1) leads to

$$\begin{aligned} &\int\limits_{\Omega} \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla v \, dx - \lambda \int\limits_{\Omega} {u_{n_k}^+}^{p^*-1} v \, dx \\ &\quad \geqslant \int\limits_{\Omega} \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla u_{n_k} \, dx - \lambda \int\limits_{\Omega} {u_{n_k}^+}^{p^*} \, dx \\ &\quad \geqslant \int\limits_{\Omega} \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla u \, dx - \lambda \int\limits_{\Omega} {u_{n_k}^+}^{p^*-1} u \, dx + o(1), \end{aligned}$$

that is,

$$\int_\Omega \mathcal{A}(x, \nabla u_{n_k}) \cdot \nabla(v-u)\,dx \geqslant \lambda \int_\Omega {u_{n_k}^+}^{p^*-1}(v-u)\,dx + o(1).$$

Passing to the limit and using Proposition 3.2, we deduce

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla(v-u)\,dx \geqslant \lambda \int_\Omega {u^+}^{p^*-1}(v-u)\,dx,$$

i.e. u is a solution of Problem 1.1.

Furthermore, if we set $v = u^+ = \max\{0, u\}$, it yields

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla(-u^-)\,dx \geqslant \lambda \int_\Omega {u^+}^{p^*-1}(-u^-)\,dx = 0.$$

We recall that $u^- = \min\{u, 0\}$. From assumption (6) of the $\mathcal{A}$-operator and

$$\nabla u^- = \begin{cases} \nabla u(x) & \text{if} \quad u(x) \leqslant 0, \\ 0 & \text{if} \quad u(x) > 0, \end{cases}$$

we have

$$0 \leqslant \alpha \int_{\Omega^-} |\nabla u^-|^p\,dx \leqslant \int_{\Omega^-} \mathcal{A}(x, \nabla u) \cdot \nabla u^-\,dx = \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla(u^-)\,dx \leqslant 0.$$

Consequently, $\nabla u^- = 0$ a.e. on Ω^-. It follows that $\nabla u^- = 0$ a.e. on Ω. Since $u^- \in W_0^{1,p}(\Omega)$, one obtains $u^- = 0$ a.e. on Ω and therefore $u \geqslant 0$ a.e. on Ω. Proposition 3.7 is proved. □

Finally, we verify that there exists a $(\mathrm{PS})_c$ sequence $\{u_n\}$ in K_ψ associated with I by Ekeland's variational principle 2.3.

Proof of Theorem 1.3. From the definition of the best Sobolev constant (2.1) and condition (2) of the variational kernel F, we get

$$\begin{aligned} I(u) &= \int_\Omega F(x, \nabla u) - \frac{\lambda}{p^*} {u^+}^{p^*}\,dx \\ &\geqslant \gamma \int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{p^*} \int_\Omega |u|^{p^*}\,dx \\ &= \int_\Omega |\nabla u|^p\,dx \left(\gamma - \frac{\lambda}{p^*} \int_\Omega |u|^{p^*}\,dx \Big/ \int_\Omega |\nabla u|^p\,dx\right) \\ &\geqslant \int_\Omega |\nabla u|^p\,dx \left(\gamma - \frac{S^{-p^*}\lambda}{p^*} \left(\int_\Omega |\nabla u|^p\,dx\right)^{\frac{p}{n-p}}\right). \end{aligned}$$

Define

$$\rho(\lambda) := \left(\frac{\gamma p^*}{2\lambda}\right)^{\frac{n-p}{p^2}} S^{\frac{n}{p}},$$

$$B_{\rho(\lambda)} := \left\{u \in W_0^{1,p}(\Omega)\colon \|u\|_{W_0^{1,p}(\Omega)} \leqslant \rho(\lambda)\right\},$$

and $\mathrm{K}_\psi(B_{\rho(\lambda)}) := \mathrm{K}_\psi \cap B_{\rho(\lambda)}$. Clearly $\rho(\lambda)$ is a decreasing function of λ, $\rho(\lambda) \to \infty$ as $\lambda \to 0$ and

$$I(u) \geqslant \frac{\gamma}{2} \int_\Omega |\nabla u|^p \, dx, \quad \forall u \in B_{\rho(\lambda)}. \tag{3.19}$$

Choose $\lambda_0 = \frac{\gamma p^* S^{p^*}}{2} (\int_\Omega |\nabla \psi^+|^p \, dx)^{-\frac{p}{n-p}}$ and then $\psi^+ \in B_{\rho(\lambda)}$ for $\lambda \in (0, \lambda_0)$. Therefore $\psi^+ \in \mathrm{K}_\psi(B_{\rho(\lambda)})$ and $I(u)$ is continuous and bounded from below in $\mathrm{K}_\psi(B_{\rho(\lambda)})$ for $\lambda \in (0, \lambda_0)$.

Next we use the perturbed minimization principle 2.3 to find a minimizing sequence $\{u_n\}$ such that

$$I(u_n) \to c := \inf\left\{I(u)\colon u \in \mathrm{K}_\psi(\overline{B}_{\rho(\lambda)})\right\},$$

which is almost critical

$$\left\langle I'(u_n), v - u_n\right\rangle \geqslant \langle z_n, v - u_n\rangle, \quad \forall v \in \mathrm{K}_\psi, \quad \text{where } z_n \to 0 \text{ as } n \to \infty.$$

By Proposition 2.3 (Ekeland), for any $n > 0$, there is a $u_n \in \mathrm{K}_\psi(B_{\rho(\lambda)})$ such that

$$c \leqslant I(u_n) \leqslant c + \frac{1}{n}, \tag{3.20}$$

$$I(w) \geqslant I(u_n) - \frac{1}{n}\left\|\nabla(w - u_n)\right\|_{L^p(\Omega)}, \quad \forall w \in \mathrm{K}_\psi(B_{\rho(\lambda)}). \tag{3.21}$$

For the sake of clarity, we only consider L^p norm of gradients although there is no difficulty to deal with $W_0^{1,p}(\Omega)$ using Poincaré's inequality. It follows from (3.19) that there exists $u \in \mathrm{K}_\psi(B_{\rho(\lambda)})$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \\ u_n &\to u \quad \text{a.e. in } \Omega. \end{aligned}$$

We claim that there exists $\lambda \in (0, \lambda_0)$ such that $\|\nabla u_n\|_{L^p(\Omega)} < \rho(\lambda)$ for n large enough.

It follows from (3.20) and assumption (2) about the variational kernel F that

$$\frac{\gamma}{2}\|\nabla u_n\|^p_{L^p(\Omega)} \leqslant I(u_n) \leqslant c + \frac{1}{n} \leqslant I\left(\psi^+\right) + \frac{1}{n} \leqslant \delta_1\left\|\nabla\psi^+\right\|^p_{L^p(\Omega)} + \frac{1}{n}.$$

We choose $\lambda \in (0, \lambda_0)$ such that

$$\rho^p(\lambda) > \frac{2\delta_1}{\gamma} \|\nabla \psi^+\|^p_{L^p(\Omega)} + 1.$$

Consequently, for n sufficiently large

$$\|\nabla u_n\|^p_{L^p(\Omega)} \leqslant \frac{2\delta_1}{\gamma} \|\nabla \psi^+\|^p_{L^p(\Omega)} + \frac{2}{n\gamma} < \rho^p(\lambda).$$

Our claim is proved.

Without loss of generality, we may assume that $\|\nabla u_n\|_{W_0^{1,p}(\Omega)} < \rho(\lambda)$ for all n.

We now proceed to prove that the sequence $\{u_n\}$ is almost critical.

In fact, for any $v \in \mathrm{K}_\psi$ and $t \in (0, 1)$, we have $u_n + t(v - u_n) \in \mathrm{K}_\psi$. For $t > 0$ small enough, we have $u_n + t(v - u_n) \in B_{\rho(\lambda)}$. So, it follows from (3.21) that

$$I\big(u_n + t(v - u_n)\big) \geqslant I(u_n) - \frac{t}{n} \big\|\nabla(v - u_n)\big\|_{L^p(\Omega)}$$

that is,

$$\frac{I(u_n + t(v - u_n)) - I(u_n)}{t} \geqslant -\frac{1}{n} \big\|\nabla(v - u_n)\big\|_{L^p(\Omega)}.$$

Letting $t \to 0^+$, we have

$$\int_\Omega \mathcal{A}(x, u_n) \cdot \nabla(v - u_n)\, dx - \lambda \int_\Omega u_n^{+\,p^*-1}(v - u_n)\, dx \geqslant \langle z_n, v - u_n\rangle,$$

where $z_n \to 0$ as $n \to \infty$. Therefore $\{u_n\}$ forms a (P.S.)$_c$ sequence for I with a subsequence converging weakly to $u \in \mathrm{K}_\psi$. Theorem 1.3 now follows from Proposition 3.7. $\square$

4. Existence of minimal positive solutions by a supersolution approach

In this section, we shall prove Theorem 1.4 by a supersolution method. The proof of this theorem is based on several lemmas. We first make the following definition.

Definition 4.1.

1. A function u in $W_0^{1,p}(\Omega)$ is a *weak solution* of the equation

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \lambda u^{+\,p^*-1} \tag{4.1}$$

in Ω if

$$\int_\Omega \mathcal{A}(x, \nabla u)\nabla\varphi\, dx = \lambda \int_\Omega u^{+\,p^*-1}\varphi\, dx$$

whenever $\varphi \in W_0^{1,p}(\Omega)$.

2. A function w in $W_0^{1,p}(\Omega)$ is a *supersolution* of (4.1) in Ω if

$$-\operatorname{div}\mathcal{A}(x,\nabla w)\geqslant \lambda w^{+p^*-1}$$

weakly in Ω, i.e.

$$\int_\Omega \mathcal{A}(x,\nabla w)\nabla\varphi\,dx\geqslant\lambda\int_\Omega w^{+p^*-1}\varphi\,dx$$

whenever a nonnegative $\varphi\in W_0^{1,p}(\Omega)$.

Since the existence of solutions relies on λ, we may attach λ to equation numbers. For example, Eq. $(1.2)_\lambda$ is the same as (1.2). Throughout this section, we assume that v is a test function in K_ψ and that φ is a nonnegative test function in $W_0^{1,p}(\Omega)$.

Heinonen, Kilpeläinen, and Martio [15, Theorem 3.23] established by the monotonicity assumption of $\mathcal{A}$-operator that the minimum of a finite number of supersolutions is also a supersolution (see also [17, Section 7]). In the following lemma, we shall prove that the infimum of any number of supersolutions in K_ψ is still a supersolution although we do not know if Eq. (4.1) is monotone.

Lemma 4.2. *Assume that $Q(\lambda)$ is the set of all supersolutions of Eq.* $(1.2)_\lambda$*, i.e.*

$$Q(\lambda):=\biggl\{w\in\mathrm{K}_\psi\colon \int_\Omega \mathcal{A}(x,\nabla w)\nabla\varphi\,dx\geqslant\lambda\int_\Omega w^{+p^*-1}\varphi\,dx,$$
$$\forall\varphi\in W_0^{1,p}(\Omega)\text{ and }\varphi\geqslant 0\biggr\},$$

then $\bar u:=\inf\{w\colon w\in Q(\lambda)\}$ is the supersolution of Eq. $(1.2)_\lambda$.

Proof. Throughout the proof of this lemma, we choose $f(x)=\lambda\bar u^{+p^*-1}$. It follows from Lemma 2.5 that the variational problem

$$\int_\Omega \mathcal{A}(x,\nabla\xi)\nabla(v-\xi)\,dx\geqslant\lambda\int_\Omega \bar u^{+p^*-1}(v-\xi)\,dx,\quad \forall v\in\mathrm{K}_\psi,$$

has a unique solution $\xi\in\mathrm{K}_\psi$.

Since $\bar u\leqslant w$ for $w\in Q(\lambda)$, we have

$$\int_\Omega \mathcal{A}(x,\nabla w)\nabla\varphi\,dx\geqslant\lambda\int_\Omega w^{+p^*-1}\varphi\,dx\geqslant\lambda\int_\Omega \bar u^{+p^*-1}\varphi\,dx$$

for all $\varphi\geqslant 0$ and $\varphi\in W_0^{1,p}$. Applying Lemma 2.6, we have

$$\xi\leqslant w.$$

Consequently

$$\xi \leqslant \bar{u}.$$

Then ξ is also a supersolution to Eq. $(1.2)_\lambda$ since by setting $v = \xi + \varphi$,

$$\int_\Omega \mathcal{A}(x, \nabla\xi)\nabla\varphi\, dx \geqslant \lambda \int_\Omega \bar{u}^{+p^*-1}\varphi\, dx \geqslant \lambda \int_\Omega \xi^{+p^*-1}\varphi\, dx.$$

Consequently

$$\bar{u} \leqslant \xi.$$

Therefore

$$\xi = \bar{u}.$$

Lemma 4.2 is proved. □

Proof of Theorem 1.4. By Theorem 1.3, there exists $\lambda^* > 0$ such that

$$\lambda^* = \sup\bigl\{\lambda > 0\colon \text{ Eq. } (1.2)_\lambda \text{ has nonnegative solutions}\bigr\}.$$

We first claim that there exists a nonnegative solution for every $\lambda \in (0, \lambda^*)$.

By Proposition 3.7, it suffices to prove that there exists a $(\mathrm{P.S.})_c$ sequence $\{u_n\}$ associated with I. In fact, by the definition of λ^*, there exists $\lambda' \in (\lambda, \lambda^*)$ such that

$$\int_\Omega \mathcal{A}(x, u) \cdot \nabla(v - u)\, dx \geqslant \lambda' \int_\Omega u^{+p^*-1}(v - u)\, dx, \quad \forall v \in \mathrm{K}_\psi,$$

has a nonnegative solution $u \in \mathrm{K}_\psi$. On the other hand, u is also a supersolution to the problem

$$\int_\Omega \mathcal{A}(x, \nabla w)\nabla\varphi\, dx \geqslant \lambda' \int_\Omega w^{+p^*-1}\varphi\, dx, \quad \forall\varphi \geqslant 0 \text{ and } \varphi \in W_0^{1,p} \tag{4.1$_{\lambda'}$}$$

in K_ψ, because, by setting $v = u + \varphi$ (clearly, $u + \varphi \in \mathrm{K}_\psi$),

$$\int_\Omega \mathcal{A}(x, \nabla u)\nabla\varphi\, dx \geqslant \lambda' \int_\Omega u^{+p^*-1}\varphi\, dx, \quad \forall\varphi \geqslant 0 \text{ and } \varphi \in W_0^{1,p}.$$

Since $\lambda' > \lambda$, we have

$$\int_\Omega \mathcal{A}(x, \nabla u)\nabla\varphi\, dx \geqslant \lambda' \int_\Omega u^{+p^*-1}\varphi\, dx \geqslant \lambda \int_\Omega u^{+p^*-1}\varphi\, dx$$

for all $\varphi \geqslant 0$ and $\varphi \in W_0^{1,p}$. Therefore u is also a supersolution to Eq. $(4.1)_\lambda$. It follows from Lemma 2.5 with $f(x) = \lambda {u^+}^{p^*-1}$ that there exists a unique solution u_1 such that

$$\int_\Omega \mathcal{A}(x, \nabla u_1)\nabla(v - u_1)\,dx \geqslant \lambda \int_\Omega {u^+}^{p^*-1}(v - u_1)\,dx, \quad \forall v \in \mathrm{K}_\psi.$$

Lemma 2.6 implies that $u_1 \leqslant u$. Consequently,

$$\int_\Omega \mathcal{A}(x, \nabla u_1)\nabla\varphi\,dx \geqslant \lambda \int_\Omega {u^+}^{p^*-1}\varphi\,dx \geqslant \lambda \int_\Omega {u_1^+}^{p^*-1}\varphi\,dx.$$

Therefore, u_1 is also a supersolution to Eq. $(4.1)_\lambda$. Using the same argument inductively, we obtain a decreasing sequence in K_ψ, $\psi \leqslant u_n \leqslant u_{n-1} \leqslant \cdots \leqslant u_1 \leqslant u$, such that

$$\int_\Omega \mathcal{A}(x, \nabla u_n)\nabla(v - u_n)\,dx \geqslant \lambda \int_\Omega {u_{n-1}^+}^{p^*-1}(v - u_n)\,dx, \quad \forall v \in \mathrm{K}_\psi. \tag{4.2}$$

Clearly $\psi^+ \leqslant u_n^+ \leqslant u_{n-1}^+ \leqslant \cdots \leqslant u_1^+ \leqslant u^+$. Therefore there exists $u_0 \in \mathrm{K}_\psi$ such that

$$u_n \to u_0 \quad \text{a.e. in } \Omega,$$
$$u_n^+ \to u_0^+ \quad \text{a.e. in } \Omega.$$

Taking $n \to \infty$ and applying Lebesgue's Dominated Convergence theorem, we get

$$\int_\Omega |u_n|^{p^*}\,dx \to \int_\Omega |u_0|^{p^*}\,dx \int_\Omega {u_{n-1}^+}^{p^*-1}(v - u_n)\,dx - \int_\Omega {u_n^+}^{p^*-1}(v - u_n)\,dx \to 0.$$

Define $z_n = {u_{n-1}^+}^{p^*-1} - {u_n^+}^{p^*-1}$. Then

$$\langle z_n, v - u_n\rangle = \int_\Omega \big({u_{n-1}^+}^{p^*-1} - {u_n^+}^{p^*-1}\big)(v - u_n)\,dx \to 0.$$

From the variational inequality (4.2), we obtain that

$$\int_\Omega \mathcal{A}(x, \nabla u_n)\cdot\nabla(v - u_n)\,dx \geqslant \lambda \int_\Omega {u_n^+}^{p^*-1}(v - u_n)\,dx + \langle z_n, v - u_n\rangle, \quad \forall v \in \mathrm{K}_\psi. \tag{4.3}$$

We set $v = u$ in (4.2) and obtain

$$\int_\Omega \mathcal{A}(x, \nabla u_n)\cdot\nabla(u - u_n)\,dx \geqslant \lambda \int_\Omega {u_{n-1}^+}^{p^*-1}(u - u_n)\,dx \geqslant 0.$$

By Hölder's inequality, conditions (6) and (7) of the $\mathcal{A}$-operator, we have

$$\alpha \int_{\Omega} |\nabla u_n|^p \, dx \leqslant \int_{\Omega} \mathcal{A}(x, \nabla u_n) \cdot \nabla u_n \, dx \leqslant \int_{\Omega} \mathcal{A}(x, \nabla u_n) \cdot \nabla u \, dx$$

$$\leqslant \beta \int_{\Omega} |\nabla u_n|^{p-1} |\nabla u| \, dx \leqslant \beta \biggl(\int_{\Omega} |\nabla u_n|^p \, dx \biggr)^{\frac{p-1}{p}} \biggl(\int_{\Omega} |\nabla u|^p \, dx \biggr)^{\frac{1}{p}}.$$

This implies

$$\|\nabla u_n\|_{L^p(\Omega)} \leqslant \frac{\beta}{\alpha} \|\nabla u\|_{L^p(\Omega)}.$$

The sequence $\{u_n\}$ is bounded in $W_0^{1,p}$. So is $\{I(u_n)\}$. Therefore there exists a subsequence and a constant c such that

$$I(u_{n_k}) \to c$$

as $k \to \infty$. It follows from (4.3) that $\{u_{n_k}\}$ is a (P.S.)$_c$ sequence. Our claim is proved. In fact, it follows by Proposition 3.7 that u_0 is a nonnegative solution of Eq. $(1.2)_\lambda$.

In our next step, we shall prove that there exists the minimal nonnegative solution to Eq. $(1.2)_\lambda$ for every $\lambda \in (0, \lambda^*)$. From Lemma 4.2, infimum $\bar{u}$ of supersolutions in K_ψ is a supersolution of variational inequality $(1.2)_\lambda$. We assert that the smallest supersolution is the minimal nonnegative solution to $(1.2)_\lambda$.

To this end, consider

$$\int_{\Omega} \mathcal{A}(x, \nabla \bar{u}_1) \nabla (v - \bar{u}_1) \, dx \geqslant \lambda \int_{\Omega} \bar{u}^{+p^*-1} (v - \bar{u}_1) \, dx, \quad \forall v \in \mathrm{K}_\psi.$$

Applying the same procedure as described earlier, we can select a (P.S.)$_c$ sequence $\{\bar{u}_n\}$ for I such that

$$\bar{u}_n \rightharpoonup \bar{u}_0 \quad \text{weakly in } W_0^{1,p} \quad \text{and} \quad \psi \leqslant \bar{u}_0 \leqslant \bar{u}.$$

$\bar{u}_0$ is also a nonnegative solution to $(1.2)_\lambda$. On the other hand, $\bar{u}_0$ is a supersolution to $(1.2)_\lambda$. Consequently

$$\bar{u}_0 \geqslant \bar{u}.$$

Therefore $\bar{u}$ must be a nonnegative solution to $(1.2)_\lambda$. Our assertion follows from the fact that all solutions to $(1.2)_\lambda$ are supersolutions of $(1.2)_\lambda$.

Finally, we show that the minimal nonnegative solution of Eq. $(1.2)_\lambda$ is a increasing function of λ. We denote the minimal nonnegative solution to $(1.2)_\lambda$ by $\bar{u}(\lambda)$. Assume that $0 < \lambda < \lambda' \in (0, \lambda^*)$. Then $\bar{u}(\lambda')$ satisfies

$$\int_{\Omega} \mathcal{A}\bigl(x, \nabla \bar{u}(\lambda')\bigr) \nabla \bigl(v - \bar{u}(\lambda')\bigr) \, dx \geqslant \lambda' \int_{\Omega} \bar{u}(\lambda')^{+p^*-1} \bigl(v - \bar{u}(\lambda')\bigr) \, dx, \quad \forall v \in \mathrm{K}_\psi.$$

On the other hand, $\bar{u}(\lambda')$ is also a supersolution of $(1.2)_{\lambda'}$, i.e.

$$\int_\Omega \mathcal{A}\big(x, \nabla\bar{u}(\lambda')\big)\nabla\varphi\, dx \geqslant \lambda' \int_\Omega \bar{u}(\lambda')^{+p^*-1}\varphi\, dx, \quad \forall\varphi \geqslant 0 \text{ and } \varphi \in W_0^{1,p}.$$

Since $\lambda < \lambda'$, we get

$$\int_\Omega \mathcal{A}\big(x, \nabla\bar{u}(\lambda')\big)\nabla\varphi\, dx \geqslant \lambda \int_\Omega \bar{u}(\lambda')^{+p^*-1}\varphi\, dx, \quad \forall\varphi \geqslant 0 \text{ and } \varphi \in W_0^{1,p}.$$

Therefore $\bar{u}(\lambda')$ is a supersolution to $(1.2)_\lambda$. The theorem follows from the fact that $\bar{u}(\lambda)$ is the smallest supersolution to $(1.2)_\lambda$. $\square$

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